Identity involving binomial coefficients with additions.
https://www.linkedin.com/feed/update/urn:li:activity:6623128692484911104
Show that for all natural numbers n, there is the identity

$$\frac{1}{\binom{2n}{1}}-\frac{1}{\binom{2n}{2}}+\frac{1}{\binom{2n}{3}}-\ldots+\frac{(-1)^{k-1}}{\binom{2n}{k}}+\ldots+\frac{1}{\binom{2n}{2n-1}}=\frac{1}{n+1}.$$

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We will consider this problem in more general setting:
Find values of the sums

$$S_n:=\sum_{k=0}^{n}\frac{(-1)^k}{\binom{n}{k}} \text{ and } T_n:=\sum_{k=0}^{n}\frac{(-1)^k(k+1)}{\binom{n}{k}}, n\in\mathbb{N}\cup\{0\}.$$

1. Calculation of S_n and T_n.

Since $(-1)^nT_n:=\sum_{k=0}^{n}\frac{(-1)^{n-k}(k+1)}{\binom{n}{k}}\overset{k=n-i}{=}\sum_{i=0}^{n}\frac{(-1)^i(n-i+1)}{\binom{n}{n-i}}\overset{k:=i}{=}\sum_{k=0}^{n}\frac{(-1)^k(n-k+1)}{\binom{n}{k}}$

then $T_n(1+(-1)^n)=\sum_{k=0}^{n}\frac{(-1)^k}{\binom{n}{k}}((k+1)+(n-k+1))=\sum_{k=0}^{n}\frac{(-1)^k(n+2)}{\binom{n}{k}}=(n+2)S_n.$

From the other hand

$$S_n=1+\sum_{k=1}^{n}\frac{(-1)^k}{\binom{n}{k}}=1+\sum_{k=0}^{n-1}\frac{(-1)^{k+1}}{\binom{n}{k+1}}=1+\sum_{k=0}^{n-1}\frac{(-1)^{k+1}(k+1)}{n\binom{n-1}{k}}=1-\frac{T_{n-1}}{n}, n\in\mathbb{N}$$

Thus, $\begin{cases} T_n(1+(-1)^n)=(n+2)S_n \\ S_n+\dfrac{T_{n-1}}{n}=1 \end{cases}, n\in\mathbb{N}.$

If n is odd, that is n then $(n+2)S_n=0\Leftrightarrow S_n=0$ and, therefore, $T_{n-1}=1-S_n=1\Leftrightarrow T_{n-1}=n, n\in\mathbb{N}$. Hence, $T_n=n+1$ if n is even;

If n is even, then $S_n=\frac{T_n(1+(-1)^n)}{(n+2)}=\frac{2(n+1)}{n+2}$ and, therefore, $T_{n-1}=n-nS_n=$ $n-\frac{2n(n+1)}{n+2}=-\frac{n^2}{n+2}$, that is $T_{2m-1}=-\frac{(2m)^2}{2m+2}=-\frac{2m^2}{m+1}$.

Thus, $S_n=\begin{cases} 0 \text{ if } n=2m-1 \\ \dfrac{2m+1}{m+1} \text{ if } n=2m \end{cases}, m\in\mathbb{N}$ and $T_n=\begin{cases} 2m \text{ if } n=2m \\ -\dfrac{2m^2}{m+1} \text{ if } n=2m-1 \end{cases}, m\in\mathbb{N}.$

2. Now we will return to the original problem.

Since $\sum_{k=0}^{2n}\frac{(-1)^k}{\binom{2n}{k}}=\frac{1}{\binom{2n}{0}}+\frac{1}{\binom{2n}{2n}}+\sum_{k=1}^{2n-1}\frac{(-1)^k}{\binom{2n}{k}}=2-\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{\binom{2n}{k}}$

and $\sum_{k=0}^{2n}\frac{(-1)^k}{\binom{2n}{k}}=\frac{2n+1}{n+1}$ then $\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{\binom{2n}{k}}=2-\frac{2n+1}{n+1}=\frac{1}{n+1}$.